Many-Particle Self-Consistent Model

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The number and general nature of the self-consistency equations that may arise in a universal bootstrap theory of all strongly interacting particles is discussed. There are more equations than there are variables to be determined, so that it may be possible to bypass some of the divergence difficulties of dispersion theory by making use of a sufficient number of the equations. A general method of attacking the difficulties associated with the many-particle aspect of the problem is discussed. A simple first approximation to the method is applied to a model of four multiplets (pseudoscalar meson, vector meson, baryon ground state, and $j=\frac{3}{2}^+$ baryon excited state), under the assumption that unitary symmetry is approximately valid. It is argued that comparison with experiment of the calculated differences in masses of particles within the same multiplet will provide experimental tests that are meaningful in a low-order approximation to the model. It is shown that if the mass splitting of the baryon octet is assumed to be partly self-generating (i.e., not resulting completely from the mass splitting of the meson multiplets), a nondegenerate solution to the model is most likely if there is a large violation of R invariance.

I. INTRODUCTION

THE universal bootstrap hypothesis of strong interaction physics is that the number of existing strongly-interacting particles, and their spins, parities, external quantum numbers, mass ratios, and interaction constants will be determined eventually from self-consistency requirements, formulated within dispersion theory. Recent successes of bootstrap calculations involving several types of particles are encouraging for this hypothesis.[1–8] Unfortunately, calculations in a model involving many types of particles are usually difficult. It is implicit in the universal bootstrap hypothesis that the equations are inconsistent if any existing particle is omitted; however, if every particle is included there are very many coupled equations. It will be necessary to discover some kind of convergent iteration procedure that is quite different from those well known in physics today, if a solution to such a model is to be obtained.

Encouraging, though incomplete, arguments have been given that the universal bootstrap hypothesis may require the approximate validity of unitary symmetry.[1,5,8] In the present paper we simply assume approximate SU_3 symmetry without theoretical justification. A type of iteration procedure for the many-particle bootstrap equations is proposed and discussed in Sec. II. It is argued in Sec. III that the signs and magnitudes of the mass differences of particles within the same SU_3 multiplet may be calculated in a very low-order approximation with sufficient accuracy to test the model. In Sec. V, the first approximation to the iteration procedure is applied to a model including the pseudoscalar meson octet, vector meson octet, ground-state baryon octet, and $j=\frac{3}{2}^+$ baryon resonance decuplet. Some of the results of this application have been obtained previously from more detailed calculations.[2–4]

II. NATURE OF THE UNIVERSAL BOOTSTRAP EQUATIONS

In a complete bootstrap model each particle occurs as a bound state or resonance pole in the coupled scattering amplitudes of the appropriate quantum numbers. It is assumed here that only two-particle scattering states need be considered, though some of the particles may be unstable. We shall attempt to count the number of self-consistency equations that will occur. We consider the pole associated with the particle X, assumed to be coupled to n two-particle states (Y_1Z_1), (Y_2Z_2), $\cdots$, (Y_nZ_n). If the spins of some of the particles are sufficiently high, there may be more than one partial-wave state associated with a particular pair of particles.

It is assumed that some dispersion-theoretic method (such as the matrix N/D method[9]) may be used to write partial-wave dispersion relations for the coupled amplitudes T_{ij} for the processes $Y_i+Z_i \rightarrow Y_j+Z_j$. The left-hand cuts (forces) for the (ij) process are associated with the singularities occurring at specific energies in the crossed processes $Y_i+\bar{Z}_j \rightarrow Y_j+\bar{Z}_i$ and $Y_i+\bar{Y}_j \rightarrow \bar{Z}_i+Z_j$. Therefore, the expression for T_{ij} is a function of the masses and interaction constants of the initial, final, and intermediate particles associated with the crossed diagrams.

For simplicity, we assume that the pole associated with the particle X is a bound-state pole at an energy below the threshold of all the two-particle states (Y_iZ_i); we will refer to an amplitude containing the pole as "resonating." (The argument may be generalized to the case of one or more open channels, although

[1] R. H. Capps, Nuovo Cimento **30**, 340 (1963).
[2] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963).
[3] R. E. Cutkosky, Ann. Phys. (N.Y.) **23**, 415 (1963).
[4] R. H. Capps, Phys. Rev. **132**, 2749 (1963).
[5] R. H. Capps, Phys. Rev. **134**, B460 (1964).
[6] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963); E. Abers, F. Zachariasen and C. Zemach, *ibid.* **132**, 1831 (1963).
[7] P. Carruthers, Phys. Rev. Letters **10**, 538 and 540 (1963); Phys. Rev. **133**, B497 (1964).
[8] Pekka Tarjanne and R. E. Cutkosky, Phys. Rev. **133**, B1292 (1964).
[9] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).

a few new complications arise in the generalization.[10]) Identification of the energy of the pole with the mass of the particle X leads to a self-consistency equation, which may be written in the form,

$$(T_{ij})^{-1}(m_X)=0, \tag{1}$$

for some of the T_{ij}. Further self-consistency equations are associated with the residues of the pole in the various amplitudes; these equations are of the form,

$$R_{ij}=-C_iC_j\gamma_i\gamma_j,$$
$$R_{ij}=\lim_{E\to m_X}[(E-m_X)T_{ij}(E)], \tag{2}$$

where γ_i is the (XY_iZ_i) interaction constant, and the C_i are known, positive functions of m_X that depend on the definition of the amplitudes T_{ij}. An equation similar to Eq. (2) may be written for each elastic and inelastic amplitude, but these equations are not all independent. In order to demonstrate this lack of independence, we note that since T is real and symmetric in the energy region of the bound state, it may be diagonalized by a real, unitary (orthogonal) transformation A. The physical amplitudes may be written in terms of the eigenamplitudes, i.e.,

$$T_{ij}=\sum_k A_{ik}A_{jk}T_{kk}. \tag{3}$$

The transformation coefficients A are energy-dependent, in general. We assume a nondegenerate bound state; the resonating eigenamplitude and its residue are denoted by T_{rr} and R_{rr}, respectively. The residue R_{rr} must be negative. The residue of T_{ij} is equal to $A_{ir}(m_X)A_{jr}(m_X)R_{rr}$, so that Eq. (2) may be written in the form,

$$A_{ir}(m_X)(-R_{rr})^{1/2}=C_i\gamma_i. \tag{4}$$

The number of independent self-consistency equations of this type is n, the number of channels coupled to the X pole.

Similar equations may be written for each type of particle. It is clear that the number of equations of the pole position type [Eq. (1)] is equal to the number of particle types, and is thus one more than the number of mass ratios. If the basic interactions are three-particle interactions, there are three self-consistency equations of the residue type [Eq. (2)] for each interaction constant. These latter equations are not all independent, however. If a particular interaction is quadratic in one type of particle (such as the $pp\pi^0$ interaction), two of the equations are identical. Some of the other equations may be dependent. For example, consider the $\rho^+\pi^+\pi^0$ interaction constant. If it can be shown that the equations require isotopic spin conservation, then the residue equations associated with the $\pi^+\to(\rho^+\pi^0)$ and $\pi^0\to(\rho^+\pi^-)$ processes will be equivalent. These equations will be different from the residue equation associated with the $\rho^+\to(\pi^+\pi^0)$ process, however. Thus, the total number of independent self-consistency equations occurring in a universal bootstrap model is greater than the number of mass ratios and interaction constants to be determined. This fact is very encouraging. It is well known that partial-wave dispersion relations are not always convergent, so that arbitrary cutoff or subtraction constants must be introduced into the equations. It is conceivable that the extra self-consistency equations may be used to eliminate these arbitrary constants, so that some of the divergence problems of dispersion theory may be bypassed.

In most of the partial bootstrap models considered so far, no use has been made of the overdetermining nature of the residue-type self-consistency equations. We consider as an example the reciprocal bootstrap model proposed by Chew.[11] In this model the N and N^* particles are associated with poles in the (1,1) and (3,3) P-wave pion-nucleon amplitudes. There are two self-consistency equations of the residue type, and the two interaction constants $\gamma_{\pi NN}$ and $\gamma_{\pi NN^*}$. In a complete bootstrap model of the π, N, and N^*, one should include the coupling of the (πN^*) channel in both these partial waves. Such an inclusion would add the one interaction constant $\gamma_{\pi N^*N^*}$ to the theory, but would lead to two more self-consistency equations of the residue type.[12] It is true that inclusion of the (πN^*) channel complicates the model, but in the long run, one may gain by this procedure if some method is found to exploit the overdetermining nature of the complete set of equations. (Of course, one must also consider the amplitude in which the pion poles develop in a complete model.)

Because of the large number of self-consistency equations, it is reasonable to suppose that all mass ratios and coupling constants are determined in a particular solution to the bootstrap equations (although there may be more than one solution). However, the number of existing (stable and unstable) strongly-interacting particles is so great that it is unlikely that a solution to a universal bootstrap model will be obtained unless some simple, convergent iterative procedure exists. The convergence of a particular iterative procedure for many equations in many unknowns depends on the assumed identification of particular equations with particular variables. (This fact is illustrated even with such a simple problem as that of solving the two linear equations, $x-2y=0$ and $2y+x=5$ by iteration, alternately solving one equation for x and the other for y.) We propose that the identification may be made in

[10] A similar argument is made for the case of two open channels in Ref. 4.

[11] G. F. Chew, Phys. Rev. Letters **9**, 277 (1962). See also F. E. Low, *ibid.* **9**, 277 (1962); J. S. Ball and D. Y. Wong, Phys. Rev. **133**, B179 (1964).

[12] This is a slight oversimplification, since there are actually two fundamental interaction constants of the πN^*N^* type. In a complete, relativistic π, N, N^* model one would have to include P-wave and F-wave (πN^*) states in the (3,3) partial wave, so that addition of the (πN^*) channel would lead to two extra interaction constants and three extra self-consistency equations.

the following, physical manner. The pole position equation associated with a particular particle is considered as the equation for the mass of that particle. The equation for the particular interaction constant γ_{abc} is formed by suitably combining the appropriate residue equations associated with the processes $a \rightarrow (bc)$, $b \rightarrow (a\bar{c})$, and $c \rightarrow (a\bar{b})$.

A slight modification of this iteration procedure does converge when applied to the nondegenerate bootstrap model of the vector-meson octet, previously considered by the author.[4] In this model, isotopic spin and hypercharge conservation are assumed, and the ρ, $M(K^*)$, and φ are considered as resonances in the various coupled two-particle states of the π, K, and η. The π, K, and η masses were taken from experiment, and the eight self-consistency equations of the model were considered as equations for $\mu_\varphi{}^2$, $\mu_\rho{}^2$, $\mu_M{}^2$, $\gamma_{\varphi KK}$, $\gamma_{\rho\pi\pi}$, $\gamma_{\rho KK}$, $\gamma_{M\pi K}$, and $\gamma_{M\eta K}$. All terms were neglected that are of power higher than the first in the deviations of the various quantities from their values in the solution in which the P and V octets are each degenerate. The self-consistency equations are linear in this approximation, and so were solved simultaneously. However, if the two equations associated with the φ pole are considered as simultaneous equations for $\mu_\varphi{}^2$ and $\gamma_{\varphi KK}$, the three ρ-pole equations are considered as simultaneous equations for $\mu_\rho{}^2$, $\gamma_{\rho\pi\pi}$, and $\gamma_{\rho KK}$, and the three M-pole equations are considered as simultaneous equations for $\mu_M{}^2$, $\gamma_{M\pi K}$, and $\gamma_{M\eta K}$, it may be shown that iteration of the three sets of equations in turn does converge to the correct answer.

More than one solution to the bootstrap equations may exist.[13] In fact, we assume the existence of at least two solutions, the physically realized solution and the "degeneracy solution," in which the SU_3 multiplets are each degenerate and the interaction constants are invariant to the SU_3 transformations. The existence of the degeneracy solution is not necessary, of course, for the success of the model. However, such a solution does exist in some simplified models of one or two SU_3 multiplets that have been considered. Furthermore, it is difficult to see how unitary symmetry can be approximately valid in a nondegenerate bootstrap model unless a degeneracy solution involving exact symmetry exists.

It is often useful to describe a solution to an approximate bootstrap model by considering the dependence of one parameter on others. This concept of dependence appears meaningless in a complete bootstrap model, since all parameters (except the absolute mass) are fixed. However, the concept may be given a meaning within the iteration procedure described above, for if attention is limited to the equations associated with a particular set of masses and coupling constants, the other constants may be considered as variable parameters within these equations. We will use this concept of dependence to discuss deviations from degeneracy within the various SU_3 multiplets. Every solution may be classified as of the σ-type (self-generating mass-splitting) or ν-type (non-self-generating mass-splitting) with respect to any set of multiplets, according to the criterion described below. One considers only the self-consistency equations associated with the masses and coupling constants of a particular set of multiplets (set I), and varies the values of the other masses and coupling constants in these equations continuously to the values of the degeneracy solution. The solution is of the ν-type with respect to set I if and only if this variation may be performed in such a way that the poles of set I exist throughout the variation, and each multiplet of set I becomes degenerate in the limit of the variation. Because of the interrelation of the particle masses in the complete model, all multiplets are expected to be nondegenerate in any solution that is of the σ-type with respect to any one or more multiplets. This type of classification of solutions is used in a simple, approximate model in Sec. V.

The σ-ν classification is related to the question of where one should start in attempting to formulate a universal bootstrap model. Although such a model is inconsistent if any type of strongly-interacting particle is neglected, the different particles may not be of equal importance in the first approximation. One procedure that may lead to a universal bootstrap model is to find a set of particles (set A) that is a subset of the existing particles, and that reproduces itself in a bootstrap model with masses and interaction constants corresponding approximately to reality. One then examines the amplitudes corresponding to all possible sets of quantum numbers to see if any bound state or resonance poles are predicted in addition to those associated with the set A. If such additional poles (set B) are found, one must redo the calculation, including the particles of set B in the input (i.e., scattering states involving the B particles, and forces transmitted by the B particles must be included). If this second calculation leads to results not greatly different from those of the first calculation, then the solution is consistent, and within the calculational scheme, the particles B are less fundamental than the particles A.[14] Clearly, it is not reasonable to investigate mass splitting in an approximate model involving only the subset A of the existing multiplets unless one believes the actual solution to be of the

[13] The idea that there may be two solutions to the same set of dynamical equations, one involving exact SU_3 symmetry and degenerate multiplets, and the other involving mass splitting, has been proposed and discussed by S. L. Glashow, Phys. Rev. **130**, 2132 (1963). It is likely that the inclusion of the electromagnetic and weak interactions in a bootstrap model would not make a great effect on calculated values of the mass ratios and strong interaction constants that occur within a particular solution. However, one can hope that these interactions "pick out" the correct (physical) solution in some manner. See Ref. 17 for a discussion of these principles as they apply to a simple model.

[14] A technique of this nature has been applied successfully to a model of the resonances of baryon number one by A. W. Martin and K. C. Wali, Bull. Am. Phys. Soc. 8, 515 (1963).

σ-type with respect to the set A. It is conceivable that the correct nondegenerate solution is of the ν-type with respect to any nontrivial subset of the existing multiplets, but if this is the case, it is difficult to imagine how this solution will ever be found. The notion that "no particles are fundamental, but some particles are more fundamental than others" is discussed further in Sec. V.

III. A MODEL OF FOUR MULTIPLETS

We assume that the basic structure of the set of strongly-interacting particles may be approximated by a bootstrap model of four SU_3 multiplets, the P (pseudoscalar meson) octet, V (vector meson) octet, B (ground-state baryon octet) and $D(j=\frac{3}{2}^+$ baryon decuplet). Several different types of two-particle configurations are coupled to each pole, i.e.,

$$\begin{aligned}\chi(P)&=(PV)+(VV)+(B\bar{B})+(B\bar{D}+D\bar{B})+(D\bar{D}),\\ \chi(V)&=(PP)+(PV)+(VV)+(B\bar{B})\\ &\qquad+(B\bar{D}+D\bar{B})+(D\bar{D}),\\ \chi(B)&=(BP)+(DP)+(BV)+(DV),\\ \chi(D)&=(BP)+(DP)+(BV)+(DV),\end{aligned}\tag{5}$$

where a bar denotes the antiparticle multiplet, and the notation $\chi(X)=(YZ)$ means that states involving one particle from the Y multiplet and one particle from the Z multiplet are coupled to the poles of the X multiplet.

No one has yet attempted to calculate in a bootstrap model involving all these configurations. One reasonable approximation procedure is to consider the coupling of each pole only with the type of configuration involving the smallest threshold energies, i.e., to consider only the first term on the right side of each of the above equations. Such a procedure has been applied to the meson multiplets by the author,[4,5] and to the baryon multiplets by Cutkosky and by Martin and Wali.[2,3] Fortunately, it is known experimentally that for each of the four multiplets, the lightest two-particle configuration is coupled strongly. However, one cannot justify convincingly the neglect of all the other configurations. For this reason, present day bootstrap calculations of such quantities as coupling constants are not reliable enough so that one can test the model by comparison with experiment.

It was pointed out in Ref. 4 that in the bootstrap model of the V octet, the neglect of all but the lightest configuration is more justifiable in calculations of mass differences of particles within the V octet than it is for the calculation of interaction constants or of the mass ratios of particles in different multiplets. This argument applies to calculations of the P, B, and D masses as well, and results from the fact that the fractional deviations from degeneracy are greatest in the P multiplet, and greater in the B than in the D multiplet. It is a striking fact that one of the important evidences for unitary symmetry, the approximate validity of the Gell-Mann–Okubo sum rule, would not be present if the symmetry were exact.[15] The author believes that the nondegeneracy of the multiplets also provides the experimental numbers that may be used in the first reliable tests of the SU_3 bootstrap model. Some such tests have already been made.[2–4]

Since the lightest configurations coupled to the V and D poles do not contain the V and D particles, we will assume that the proper solution of the model is of the ν-type (see Sec. II) with respect to the V multiplet and also with respect to the D multiplet. Different assumptions with regard to the P and B multiplets are discussed in Sec. V.

IV. THE PROBABILITY MATRIX APPROXIMATION

In this section we extend the approximation technique introduced in Ref. 5, in connection with a bootstrap model of the P and V mesons. For illustrative purposes, we first consider a simplified model involving only the π and ρ charge triplets and the $\rho\pi\pi$ interaction. In this model the ρ is a $(\pi\pi)$ resonance, and the π is a $(\pi\rho)$ bound state; we are not concerned with the nature of the forces that produce the poles. There are four independent masses, $\mu_{\pi^+}=\mu_{\pi^-}$, μ_{π^0}, $\mu_{\rho^+}=\mu_{\rho^-}$ and μ_{ρ^0}, and two independent interaction constants, $\gamma_{\rho^+\pi^+\pi^0}=\gamma_{\rho^-\pi^-\pi^0}$ and $\gamma_{\rho^0\pi^+\pi^-}$. We are interested in the possibility of a solution in which the pion triplet is not degenerate.

We will not write dispersion relations, but instead will discuss a general type of approximation. Since the ρ particle is not contained in $\chi(\rho)$, while the π is contained in $\chi(\pi)$, we assume the solution is of the ν-type for the ρ and of the σ-type for the pions, i.e., the $\pi^\pm-\pi^0$ mass difference is self-generating. It is assumed that the self-consistency equations may be combined in such a way that the interaction constants and ρ masses may be eliminated from the pion mass equations. Since absolute masses cannot be calculated, the pion mass equation may then be written in the function form, $f(\delta)=0$, where $\delta=(\mu_{\pi^+}{}^2-\mu_{\pi^0}{}^2)/(\frac{2}{3}\mu_{\pi^+}{}^2+\frac{1}{3}\mu_{\pi^0}{}^2)$. The degeneracy solution $\delta=0$ is presumed to exist, but another solution is desired. If a second solution for δ is found, the $\rho^+-\rho^0$ mass splitting may be calculated in terms of δ.

Partial wave dispersion relations are sufficiently complicated so that it may be quite difficult to compute the function $f(\delta)$. It may be easier to compute the coefficients of the first few terms in a power-series expansion, i.e.,

$$f(\delta)=c\delta+d\delta^2+\cdots=0,$$

where c and d are constants. The value $\delta=-c/d$ is a solution to the second-order equation. This second-order solution exists even if no nondegenerate solution to $f(\delta)=0$ actually exists. Therefore, this approximation is not sufficiently accurate for an investigation of the

[15] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

question of the existence of a nondegenerate solution. However, if d is nonzero, the solution $\delta=-c/d$ will approximate a real solution if $|c|$ is small enough.[16]

A similar argument applies if there are three independent P-meson masses. In this case there are two independent mass-splitting variables, δ_1 and δ_2. If the self-consistency equations are expanded to second order in the δ_i, and the linear terms in the equations are diagonalized, the equations are of the form,

$$c_1\delta_1+f_1(\delta_1,\delta_2)=0,$$
$$c_2\delta_2+f_2(\delta_1,\delta_2)=0,$$

where f_1 and f_2 are of second order. We assume these equations are normalized in some physical manner. The ratio δ_1/δ_2 is given by $\delta_1/\delta_2=(c_2/c_1)(f_1/f_2)$. Thus, a nondegenerate solution involving an appreciable value of δ_i is most likely if $|c_i|$ is small. We conclude that a first attack on an actual problem involving several possible types of mass splitting may be made by computing the coefficients of the terms of the self-consistency equations that are linear in the mass differences, diagonalizing the equations, and looking for a small coefficient. This technique is applied in Ref. 5 and in Sec. V of the present paper. A similar technique is applied to a model of vector mesons by Cutkosky and Tarjanne.[17]

We now return to the SU_3 model of Sec. III and consider, for definiteness, the equations for the masses and interaction constants of the members of the B octet. Some of the important features of nondegenerate solutions may be examined in the "mass approximation," defined as the approximation in which the particle masses, but not the interaction constants, are allowed to vary from the degeneracy-solution values. Only the pole-position self-consistency equations are considered; these equations are identified with the masses of the particles corresponding to the poles, as discussed in Sec. II.

We make a further approximation by neglecting the effect of deviations from degeneracy on the left-hand cut. In this approximation, deviations of the masses of the particles in the two-particle states coupled to the baryon poles influence the pole positions because the integrands in the dispersion integrals associated with the right-hand cut contain powers of the center-of-mass momenta of the particles in the various two-particle states. The magnitude of this momentum for a particular state is given by the formula,

$$4q^2=s-2(\mu_1^2+\mu_2^2)+(\mu_1^2-\mu_2^2)^2/s, \qquad (6)$$

where μ_1 and μ_2 are the masses of the two particles, and s is the square of the total energy. Since only the squares of the masses and the squares of the total energy appear in this equation, we will use these quantities as basic variables. This seems to conflict with intuitive notions regarding the importance of the thresholds of the various channels, since the position of a threshold is a linear sum of the appropriate masses. However, it must be remembered that it is the integral of the discontinuity across a branch cut, rather than the position of the end of the cut, that is important. This integral is a function of the squares of the masses.

We consider the coupling of the B poles only to two-particle states of the (PB) type. Since the effects of deviations from degeneracy on the left-hand cut are neglected, the only variables in the problem are the masses of the various P mesons and baryons. The symbols m_i and μ_j are used to denote the masses of the baryon i and P meson j. The fractional deviations in the squares of these masses are denoted by Δ_i and δ_j, i.e., $\Delta_i=(m_i^2-m_0^2)m_0^{-2}$ and $\delta_j=(\mu_j^2-\mu_0^2)\mu_0^{-2}$, where m_0 and μ_0 are a pair of values that satisfy the degeneracy solution. The self-consistency equation associated with the position of the pole of the baryon i is the equation for Δ_i. As in the π-ρ example discussed above, we expand the quantities in this equation in powers of the mass deviations, i.e.,

$$\Delta=\sum_j \alpha_B\Pi_{B,ij}\Delta_j+\sum_k \alpha_P\Pi_{P,ik}\delta_k+O_{2,i}, \qquad (7)$$

where $O_{2,i}$ represents the terms of second order in the δ and Δ, and α_B, α_P, $\Pi_{B,ij}$, and $\Pi_{P,ij}$ are constants, α_P and α_B being chosen for convenience.

It is shown in the Appendix that in a simple model possessing the assumptions of the present section, the quantities $\Delta_{B,ij}$ and $\Delta_{P,ik}$ may be set equal to the probabilities of the baryon j and of the P meson k in the degeneracy-solution wave function for the baryon i. Explicitly, if the degeneracy-solution wave function is $\chi(B_i)=\sum_{lm}C_{ilm}B_lP_m$, then $\Pi_{B,ij}=\sum_m(C_{ijm}^2)$ and $\Pi_{P,ik}=\sum_l(C_{ilk}^2)$. It is also shown that α_B and α_P are both positive if these substitutions for the Π factors are made.

In Sec. V, m_0^2 and μ_0^2 will be set equal to the average experimental values of the m_i^2 and μ_i^2, so that $\sum_i\Delta_i=\sum_i\delta_i=0$. It is not necessary to make this interpretation, however; one may regard the Δ_i and δ_i as all independent. In fact, since absolute masses cannot be determined by means of the dispersion relations, the substitution of $\Delta_i=\Delta_j=\delta_i=\delta_j=\cdots$ into Eq. (7) must lead to a continuum of degeneracy solutions. This condition implies the relation,

$$\alpha_B+\alpha_P=1. \qquad (8)$$

The Δ_i, δ_i, $O_{2,i}$, $\Pi_{B,ij}$, and $\Pi_{P,ik}$ in Eq. (7) may be considered as components of column vectors $\boldsymbol{\delta}$, $\boldsymbol{\Delta}$, and $\mathbf{O}_2$, and rectangular matrices Π_B and Π_P. The matrix equation for $\boldsymbol{\Delta}$ is

$$\boldsymbol{\Delta}=(1-\alpha_P)\Pi_B\boldsymbol{\Delta}+\alpha_P\Pi_P\boldsymbol{\delta}+\mathbf{O}_2. \qquad (9)$$

[16] The application of the probability matrix approximation to the π-ρ system in Ref. 5 suggests that a nondegenerate solution does not exist.

[17] R. E. Cutkosky and Pekka Tarjanne, Phys. Rev. **132**, 1354 (1963).

Since α_B and α_P are both positive, α_P is in the range 0–1. We refer to Eq. (9), together with the condition $0<\alpha_P<1$, as the "probability matrix approximation."

V. APPLICATION TO FOUR-MULTIPLET MODEL

A. The Baryon Octet

An approximate solution to the (BP) model of the baryon octet has been given by Cutkosky, and by Tarjanne and Cutkosky.[3,8] In the terminology of Sec. II of the present paper, this solution is of the ν-type, with respect to the B multiplet. In the present section, the ν and σ solutions to this model will be compared, and the relation between the σ solution and R invariance discussed.

If only the (BP) states are included in $\chi(B)$, the equations for the baryon mass splitting in the probability matrix approximation are given by Eq. (9). As discussed in Sec. IV, the elements of the probability matrices Π_B and Π_P are determined from the coefficients in the degeneracy-solution wave functions for the $\chi(B)$. The coefficients are proportional to the BBP interaction constants. In a complete model they are determined from the bootstrap equations. There is evidence from special examples that the consistency conditions require SU_3 symmetry in the degeneracy solution.[18] In the present paper we do not examine the left-hand cut in order to write self-consistency equations, but simply assume that a self-consistent degeneracy solution exists, and that this solution possesses SU_3 symmetry. The coefficients in the equation for $\chi(B)$ may be determined by substituting ordinary isotopic-spin Clebsch-Gordon coefficients into Eq. (16) of Ref. 3. We give below the equation for $\chi(p)$ as an illustration.

$$\begin{aligned}2\chi(p)=&(p\pi^0)(f+3^{1/2}d)+(n\pi^+)(2^{1/2}f+6^{1/2}d)\\&+(p\eta)(3^{1/2}f-d)+(\Sigma^+K^0)(-2^{1/2}f+6^{1/2}d)\\&+(\Sigma^0K^+)(-f+3^{1/2}d)+(\Lambda K^+)(-3^{1/2}f-d),\end{aligned}$$

where f and d are given in terms of the interaction angle θ by the relations,

$$f=3^{-1/2}\sin\theta, \quad d=5^{-1/2}\cos\theta. \tag{10}$$

In order to investigate the solutions of Eq. (9) it is convenient to express $\mathbf{\Delta}$ in terms of the eigenvectors of Π_B. These eigenvectors have been listed previously by the author.[19] The normalized vectors $\mathbf{E}_0$, $\mathbf{E}_3$, $\mathbf{E}_4$, and $\mathbf{E}_7$ of Ref. 19 are eigenvectors for all values of the interaction angle θ. Two pairs of θ-dependent eigenvectors exist in the $\mathbf{E}_1$-$\mathbf{E}_2$ and $\mathbf{E}_5$-$\mathbf{E}_6$ subspaces. We define Δ_i and $\mathbf{\Delta}_i$ (i ranges from 1 to 7) by the equations $\Delta_i=(1/8)^{1/2}(\mathbf{\Delta}\cdot\mathbf{E}_i)$ and $\mathbf{\Delta}_i=\Delta_i\mathbf{E}_i$. For convenience, we list below the formulas for Δ_1, Δ_2 and Δ_3, the isotopic-spin conserving components of the mass-splitting vector,

$$\Delta_1=(1/160)^{1/2}(p+n+\Xi^-+\Xi^0+2\Lambda-2\Sigma^+-2\Sigma^--2\Sigma^0),$$

$$\Delta_2=(1/32)^{1/2}(p+n-\Xi^--\Xi^0),$$

$$\Delta_3=(1/960)^{1/2}(\Sigma^++\Sigma^-+\Sigma^0+9\Lambda-3p-3n-3\Xi^--3\Xi^0).$$

The symbol for the baryon j is used to represent $(m_j^2-m_0^2)/m_0^2$, where m_0^2 now is set equal to the average square of the eight baryon masses. An analogous definition is made for the P-meson mass vector $\boldsymbol{\delta}$. The expression for δ_i is obtained from the corresponding expression for Δ_i by making the replacements $m_j^2\rightarrow\mu_j^2$ and $m_0^2\rightarrow\mu_0^2$, where μ_0^2 is the average square of the meson masses and a particular j relates corresponding members of the B and P octets.[20] The equality of the masses of a particle and antiparticle implies that δ_2 (as well as the isotopic spin violating components δ_6 and δ_7) is zero. The Okubo sum rule is the statement all the δ_i and Δ_i except δ_1, Δ_1, and Δ_2 should be zero.

We write the baryon mass-splitting vector $\mathbf{\Delta}$ as a sum of two parts, i.e., $\mathbf{\Delta}=\mathbf{\Delta}_\nu+\mathbf{\Delta}_\sigma$. The term $\mathbf{\Delta}_\nu$ is defined as the solution to Eq. (9) when the second-order term $\mathbf{O}_2$ is neglected. Thus, $\mathbf{\Delta}_\nu$ represents the linear approximation to that part of the B mass splitting that results from the P mass splitting. In the terminology of Sec. II, $\mathbf{\Delta}$ represents a ν-type solution if $\mathbf{\Delta}_\sigma$ is zero, and a σ-type solution if $\mathbf{\Delta}_\sigma$ is nonzero. The $\mathbf{\Delta}_\nu$ and $\mathbf{\Delta}_\sigma$ also will be expressed in terms of the eigenvectors of Π_B.

It is shown in Ref. 3 that in a linear approximation to the ν-type solution, the Okubo sum rule for the P mesons leads to the Okubo sum rule for the baryons. We neglect all contributions to the experimental $\boldsymbol{\delta}$ vector except the large Okubo-type contribution $\boldsymbol{\delta}_1$. We may then neglect all components of $\mathbf{\Delta}_\nu$ except $\Delta_{1,\nu}$ and $\Delta_{2,\nu}$. The linear approximation to the ν-type solution is,

$$\begin{aligned}\Delta_{1,\nu}&=\tfrac{3}{2}(f^2-d^2)(\Delta_{1,\nu}+\delta_1)+(15)^{1/2}df\Delta_{2,\nu}\\\Delta_{2,\nu}&=(15)^{1/2}df(\Delta_{1,\nu}-\delta_1)+\tfrac{1}{2}\Delta_{2,\nu}.\end{aligned} \tag{11}$$

If $\mathbf{\Delta}=\mathbf{\Delta}_\nu+\mathbf{\Delta}_\sigma$ is substituted into Eq. (9), the $\mathbf{\Delta}_\nu$ and $\boldsymbol{\delta}$ parts cancel and the equation for $\mathbf{\Delta}_\sigma$ becomes,

$$[1-(1-\alpha_P)\Pi_B]\mathbf{\Delta}_\sigma=\mathbf{O}_2. \tag{12}$$

We write $\mathbf{\Delta}_\sigma$ as a sum of eigenvectors of Π_B; i.e., $\mathbf{\Delta}_\sigma=\sum_i\mathbf{\Delta}_{i,\sigma}$; $\Pi_B\mathbf{\Delta}_{i,\sigma}=a_i\mathbf{\Delta}_{i,\sigma}$. Equation (12) then becomes

$$\sum_i c_i\mathbf{\Delta}_{i,\sigma}=\mathbf{O}_2, \tag{13}$$

$$c_i=1-(1-\alpha_P)a_i. \tag{14}$$

The eigenvalues a_i are easily found. They are given

[18] R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).

[19] R. H. Capps, Phys. Rev. **134**, B460 (1964).

[20] The quantities Δ_i and δ_i (i from 1 to 7) are related to the quantities of Table I of Ref. 19 by the relations,

$$\Delta_i=\mathbf{m}_B{}^2\cdot\mathbf{E}_i/\mathbf{m}_B{}^2\cdot\mathbf{E}_0 \quad\text{and}\quad \delta_i=\mathbf{m}_P{}^2\cdot\mathbf{E}_i/\mathbf{m}_P{}^2\cdot\mathbf{E}_0.$$

in terms of the interaction angle θ by the equations,

$$a_3=a_4=d^2-f^2,$$
$$a_7=-2d^2,$$
$$a_{12}^{\pm}=\tfrac{1}{2}d^2+\tfrac{3}{2}f^2\pm(4d^4+15d^2f^2)^{1/2},$$
$$a_{56}^{\pm}=d^2+f^2\pm\tfrac{1}{2}(f^4+9d^4+42d^2f^2)^{1/2}, \qquad (15)$$

where d and f are given in Eq. (10).

The principle discussed in connection with the π-ρ example of Sec. IV implies that a σ-type solution involving a large value of $|\Delta_{i,\sigma}|$ is most likely if the coefficient c_i of Eq. (13) is small. It may be shown from Eqs. (15) and (10) that for all values of θ, all the eigenvalues a_i associated with mass-splitting are smaller than one. It is seen from Eq. (14) that the conditions $a_i<1$ and $0<\alpha_P<1$ imply that $c_i>0$, and that the smallest value of c_i corresponds to the largest positive value of a_i. For simplicity, we assume that $\mathbf{\Delta}_\sigma$ is in the direction corresponding to the largest possible positive eigenvalue of any a_i. It may be seen from an examination of Eq. (15) that the largest possible eigenvalue is $a_{12}^{+}=6/7$, this value occurring when $\theta=\pm\tan^{-1}(9/5)^{1/2}\sim\pm53°$. We set $\theta=53°$, the positive sign being chosen to agree with the experimental fact that the $j=\frac{3}{2}^+$ decuplet occurs in the representation 10 rather than 10*.[2,21] The predicted ratio $\Delta_{1,\sigma}/\Delta_{2,\sigma}$ then is equal to $(5/9)^{1/2}$. [The ratio $\Delta_{1,\sigma}/\Delta_{2,\sigma}$ may be determined for any θ, if use is made of the coefficients of $\Delta_{1,\nu}$ and $\Delta_{2,\nu}$ in Eq. (11), as these coefficients are the elements of Π_B in the $\mathbf{E}_1$-$\mathbf{E}_2$ subspace.]

In order to compare the ν-type and σ-type solutions, we must estimate the value of α_P, the parameter measuring the relative importance of P and B mass deviations for the position of the B poles. In this probability matrix approximation, the effect of increasing the mass of one of the particles in one of the (BP) states is to decrease the momentum in the dispersion integral corresponding to that state. The quantity $\alpha_P(1-\alpha_P)^{-1}$ would be equal to the ratio

$$\left(\mu_0^2\frac{\partial q^2}{\partial\mu_0^2}\right)\Big/\left(m_0^2\frac{\partial q^2}{\partial m_0^2}\right)$$

if this ratio were constant throughout the energy region of the dispersion integrals [see Eq. (A6) of the Appendix]. Actually, as may be seen from Eq. (6), this ratio varies monotonically with energy, being equal to (μ_0/m_0) at threshold, and approaching (μ_0^2/m_0^2) at the high-energy limit. Therefore, we assume that

$$(\mu_0^2/m_0^2)<\alpha_P/(1-\alpha_P)<(\mu_0/m_0). \qquad (16)$$

If μ_0^2 and m_0^2 are set equal to the average values of the squares of the P and B masses, Eq. (16) leads to the condition, $0.11<\alpha_P<0.26$. A reasonable estimate of α_P is about 0.15 or 0.20.

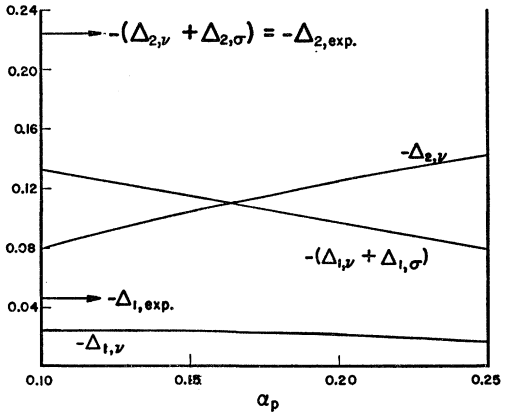

FIG. 1. Okubo-type mass splittings for the baryon octet. The calculated Δ's correspond to $\theta=\tan^{-1}(9/5)^{1/2}$.

The values of $\Delta_{1,\nu}$ and $\Delta_{2,\nu}$ corresponding to α_P in the range. $0.1-0.25$ and to $\theta=\tan^{-1}(9/5)^{1/2}$ are compared to the experimental values $\Delta_{1,\text{exp}}$ and $\Delta_{2,\text{exp}}$ in Fig. 1. The magnitude of the $\Delta_{2,\sigma}$ that occurs in the self-generating type of solution cannot be determined if only first-order terms are calculated. Hence this value has been chosen so that the total Δ_2 agrees with experiment. The value $\Delta_{1,\sigma}$ may then be determined from the equation $\Delta_{1,\sigma}=(5/9)^{1/2}\Delta_{2,\sigma}$; the calculated sum $\Delta_1=\Delta_{1,\nu}+\Delta_{1,\sigma}$ is also compared with experiment in Fig. 1. The experimental values of δ (used in the calculation of the Δ_ν) and of the Δ_1 and Δ_2 are taken from Ref. 19, and correspond to the compilation of Roos.[22]

It is seen from Fig. 1 that in the ν solution, the predicted signs of Δ_1 and Δ_2 agree with experiment, and the predicted ratio Δ_1/Δ_2 is close to the experimental value.[23] However, the magnitudes of $\Delta_{1,\nu}$ and $\Delta_{2,\nu}$ are only about half the experimental values. This is consistent with the assumption that the σ solution corresponds to reality. In the present approximation to the σ solution, the sign of Δ_2 cannot be determined, but the predicted Δ_1/Δ_2 ratio is positive and less than one, though somewhat larger than the experimental value of 0.21. The calculation is sufficiently crude that we can only conclude that either solution is consistent with experiment.

The eigenvalue a_{12}^{+} is plotted against θ in Fig. 2. It is seen that the peak is fairly broad, so that the requirement of large a_{12}^{+} does not fix θ very precisely. It has been shown by the author, and by Martin and Wali, that the experimental observation that the decuplet is the most attractive $j=\frac{3}{2}^+$ state of the (BP) type limits the range of the angle θ.[2,21] The careful calculation of

[21] R. H. Capps, Nuovo Cimento **27**, 1208 (1963).

[22] Matts Roos, Rev. Mod. Phys. **35**, 314 (1963).

[23] These facts are pointed out by Cutkosky in Ref. 3, where an extensive discussion of the ν-type solution is contained. See also Ref. 8.

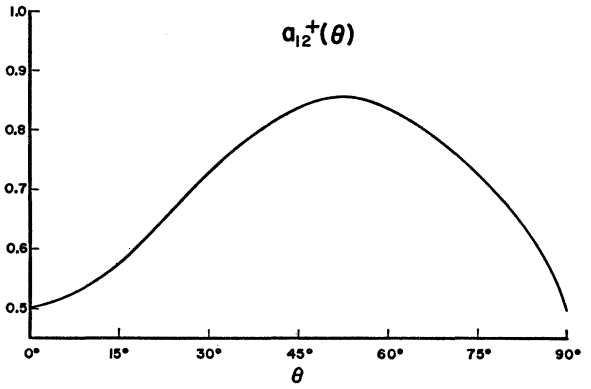

FIG. 2. The large eigenvalue a_{12}^+ for Okubo-type mass splitting of the baryon octet, as a function of θ.

Martin and Wali leads to the limits $13°<\theta<60°$. Glashow and Rosenfeld have used experimental data to make the estimate $\theta\sim35°$.[24] It is seen from Fig. 2 that at $\theta=35°$ the value of a_{12}^+ is almost as large as the peak value $[a_{12}^+(35°)\sim0.77]$. If $\theta=35°$, then the ratio $\Delta_{1,\sigma}/\Delta_{2,\sigma}$ is 0.58, and a calculation of curves similar to those of Fig. 1 leads to essentially the same conclusions concerning the ν and σ solutions.

In Sec. V of Ref. 5, considerations similar to the above were applied to the P octet, under the assumption that the P may be represented as (PV) compounds. Self-generated mass differences of the Okubo type are favored for this multiplet also. The probability matrix approximation leads to an equation for the δ_i similar to Eq. (13), with the constant c (for Okubo-type mass-splitting) given by the formula

$$c=1-\tfrac{1}{2}(1-\alpha_{PV})-\tfrac{1}{4}\alpha_{PV}, \qquad (17)$$

where $0<\alpha_{PV}<1$, and the ratio $\alpha_{PV}/(1-\alpha_{PV})$ measures the relative importance of V and P mass splitting in the (PV) states. Since this constant c is greater than $\frac{1}{2}$, while c_{12}^+ for the baryon octet may be as small as 0.2, the assumption that both the P and B mass splittings are self-generating is consistent with the experimental fact that the relative splittings are much greater within the P octet.

The principal difference between the ν and σ solutions for the baryon octet concerns the interaction angle θ. The present investigation suggests that the very existence of a σ solution may depend on a large violation of R symmetry. It has been pointed out in Ref. 5 that the requirement of self-generating mass differences may rule out many group-representation schemes for strongly-interacting particles completely, as well as favor a particular type of mass splitting within a particular scheme. It is conceivable that the large value of a_{12}^+ that is possible for the baryon multiplet within the octet scheme of SU_3 is one of the reasons this scheme is realized in nature.[25]

[24] S. L. Glashow and A. H. Rosenfeld, Phys. Rev. Letters **10**, 192 (1963).

B. Remarks Concerning V, D, and P Multiplets

In this section we apply the probability matrix approximation to the V and D multiplets, and also consider the possibility that the mass differences within the P octet result from the mass differences within the B octet. Detailed calculations concerning the D and V multiplets have been given previously[2–4,8]; we consider these multiplets here only to test the validity of the probability matrix approximation. All isotopic spin violating contributions to mass splitting are neglected.

Only the (PP) contribution is considered in the equations for the V poles. We assume that the mass splitting of the V octet (ρ, $M=K^*$, φ) is of the ν-type, and results from the P mass splitting. Calculation of the V mass splitting is similar to the calculation of the ν solution in Sec. VA; only linear terms need be considered in order that an approximate answer be obtained. The degeneracy-solution wave functions may be obtained by making appropriate $P\rightleftarrows V$ substitutions in Eq. (4) of Ref. 1. Since only one type of particle appears in the (PP) configurations, there is no unknown α parameter; the probability matrix approximation leads to the prediction,

$$\delta_1{}^V=\tfrac{1}{2}\delta_1, \quad \delta_3{}^V=-\tfrac{1}{3}\delta_3, \qquad (18)$$

where the δ^V refer to the V octet, and are defined analogously to the δ of the P octet. It is seen that in this approximation, an Okubo-type P mass splitting leads to an Okubo-type V mass splitting. The calculated values of the ratios $(m_\rho{}^2/m_M{}^2)$ and $(m_\varphi{}^2/m_M{}^2)$ that follow from the experimental P masses are shown in column 2 of Table I. The corresponding experimental values are given in column 1. [The ambiguity in the experimental $(m_\varphi{}^2/m_M{}^2)$ ratio results from the ambiguity in the interpretation of the experimental ω and φ.] Column 3 contains the results of the detailed calculation of Ref. 4; in this reference the effects of deviations from degeneracy in the virtual V multiplet that is assumed

TABLE I. Vector meson mass ratios.

	(1) Experiment	(2) Prob. matrix	(3) Ref. 4
$\mu_\rho{}^2/\mu_M{}^2$	0.72	0.47	0.30–0.72
$\mu_\varphi{}^2/\mu_M{}^2$	0.79–1.33	1.21	0.98–1.11

[25] To the author's knowledge, the eigenvalues a_i occurring in the (BP) model of the baryons have not been calculated for group-representation schemes other than the double-octet scheme of SU_3. However, similar constants occurring in the (PV) model of the P mesons are evaluated for many group-representation schemes in Ref. 5. None of these eigenvalues calculated for the (PV) model is as large as 6/7.

to transmit the forces, and the effects of deviations from SU_3 symmetry of the interaction constants, are taken into account. It is seen that the probability matrix approximation reproduces the most important effects of the detailed calculation.

We now consider the decuplet poles, neglecting all coupled configurations except the (BP) configurations. The Okubo sum rule is simple for the decuplet, i.e., $\Omega-\Xi^*=\Xi^*-Y^*=Y^*-N^*$, where the particle symbols refer to the corresponding masses or squares of masses.[15] This sum rule, when applied to the known members of the decuplet (Ξ^*, Y^*, and N^*), is satisfied more nearly by the masses than by the squares of the masses. If the fractional deviations of the squares are denoted by D, the experimental masses listed in Ref. 22 lead to the results

$$D_{\Xi Y}=(m_{\Xi^*}{}^2-m_{Y^*}{}^2)/m_{Y^*}{}^2=0.226\,,$$
$$D_{YN}=(m_{Y^*}{}^2-m_{N^*}{}^2)/m_{Y^*}{}^2=0.202\,.$$

In view of the difficulty of associating definite masses with resonances, we do not regard the difference between these two numbers as significant evidence that the squares of the masses are not appropriate variables. We take the average of $D_{\Xi Y}$ and D_{YN} as the experimental value of the mass splitting, i.e., $D_{\text{exp}}=0.214$.

It is assumed that the decuplet mass splitting is of the ν-type, and results from the P and B mass splittings. Martin and Wali and Wali and Warnock have shown that the experimental decuplet masses may be reproduced from a bootstrap model in which $\chi(D)=(BP)$, and Cutkosky has shown that the Okubo sum rule for the decuplet follows from its assumed validity for the P and B octets.[2,3,26] Hence we test the validity of the probability matrix approximation by calculating D and comparing it with the experimental value. The elements of the relevant probability matrices may be determined from the decuplet wave functions given by Glashow and Sakurai.[27] It is easily shown that in this approximation the magnitude of D is given by the expression

$$(10)^{1/2}D=(1-\alpha_{DP})(\Delta_1-5^{1/2}\Delta_2)+\alpha_{DP}\delta_1\,, \quad (19)$$

where $0<\alpha_{DP}<1$, and the ratio $\alpha_{DP}/(1-\alpha_{DP})$ measures the relative importance of P and B mass deviations. If the values of the Δ_i and δ_1 listed in Ref. 19 are substituted into this expression, the result is

$$D=0.145(1-\alpha_{DP})+0.220\alpha_{DP}\,. \quad (20)$$

This result is not a sensitive function of α_{DP}. The reasonable estimate $\alpha_{DP}\sim 0.2$ leads to $D\sim 0.16$, which is about $\frac{3}{4}$ the experimental value of 0.214. We conclude that the probability matrix approximation is fairly accurate for the decuplet.

Next we consider the P meson poles. Two bootstrap models that have been proposed for the P mesons are that they are (PV) states, on the one hand,[1,5] or $(B\bar{B})$ states on the other.[28] It must be emphasized that both of these models are approximations; since the $\rho\pi\pi$ and $NN\pi$ interaction constants are both known to be large, there is no question about the fact that (PV) and $(B\bar{B})$ states both play appreciable roles in the P-meson wave functions. However, it may be justifiable to neglect one of these two configurations when discussing certain of the P-meson properties. If the P mass splitting is of the σ-type with respect to the meson multiplets, it is reasonable to neglect the $(B\bar{B})$ states in the first approximation; this approach is followed in Ref. 5. At present, we wish to make the alternate assumption that the P mass splitting is of the ν-type, and results from the baryon mass splitting. We write the P-meson, degeneracy-solution wave function as a sum of two contributions, i.e., $\chi(P)=\chi_V+\chi_B$, where χ_V and χ_B represent the (PV) and $(B\bar{B})$ contributions. We take the PBB interaction angle to be $\theta=\tan^{-1}(9/5)^{1/2}$, since this choice corresponds to the vanishing of the $\pi\Xi\Xi$ interaction, and is thus particularly favorable for the hypothesis that the low π-mass results from the low nucleon mass. The expressions for χ_B are,

$$\chi(\pi)=(3/7)^{1/2}(N\bar{N})+(1/14)^{1/2}[(\Lambda\bar{\Sigma})+(\Sigma\bar{\Lambda})]+(3/7)^{1/2}(\Sigma\bar{\Sigma})\,,$$
$$\chi(\eta)=(1/7)^{1/2}(N\bar{N})+(1/14)^{1/2}(\Lambda\bar{\Lambda})+(3/14)^{1/2}(\Sigma\bar{\Sigma})+(4/7)^{1/2}(\Xi\bar{\Xi})\,,$$
$$\chi(K)=(2/7)^{1/2}(N\bar{\Lambda})+(1/14)^{1/2}(\Lambda\bar{\Xi})+(9/14)^{1/2}(\Sigma\bar{\Xi})\,. \quad (21)$$

It is easily shown from the probability matrix corresponding to Eq. (21) that B deviations of the Okubo types Δ_1 and Δ_2 both contribute only to the Okubo-type P deviation δ_1. If use is made of Eqs. (17) and (21) and the condition that absolute masses may not be calculated, the equation for δ_1 in the probability matrix approximation may be written in the form,

$$\delta_1=(1-\beta)\lambda\delta_1+(3/14)\beta(\Delta_1-5^{1/2}\Delta_2)\,,$$
$$\lambda=\tfrac{1}{2}(1-\alpha_{PV})+\tfrac{1}{4}\alpha_{PV}\,, \quad (22)$$

where α_{PV} and β are positive numbers smaller than unity. The constant β depends on the relative importance of the $(B\bar{B})$ and (PV) configurations.

The value of δ_1 that results from substitution of the experimental Δ_1 and Δ_2 into Eq. (22) is the linear approximation to the ν-type solution for δ_1. It is easy to see that $|\delta_1|$ is a monotonically increasing function of β, so that the upper limit for $|\delta_1|$ corresponds to $\beta=1$. If Δ_1 and Δ_2 are taken from Ref. 19, this upper limit is only about 0.14 of the experimental δ_1. We conclude that the P mass splitting is not a likely result of the B mass splitting.

[26] K. C. Wali and R. L. Warnock, Bull. Am. Phys. Soc. **9**, 115 (1964), and (private communication).

[27] S. L. Glashow and J. J. Sakurai, Nuovo Cimento **25**, 337 (1962).

[28] C. N. Yang and E. Fermi, Phys. Rev. **76**, 1739 (1946); Y. Miyamoto, Progr. Theoret. Phys. (Kyoto) **28**, 967 (1962); Yasuo Hara, Phys. Rev. **133**, B1565 (1964).

It is sometimes remarked that in the $(B\bar{B})$ model of the P mesons, the ratios of the binding energies of the π, K, and η do not differ greatly from unity. From this point of view, it would not seem difficult to construct a bootstrap model in which the large P mass differences are a result of the B mass differences. However, ratios of binding energies are not very significant quantities in dispersion relations. Ratios of masses are significant quantities, but in terms of such ratios the π and η masses are greatly different in any model. The author believes that any realistic estimate of the possible effect of the B mass differences on the P mass differences must take into account explicitly the fact that absolute masses can not be determined by means of the dispersion equations.

VI. CONCLUDING REMARKS

It is shown in Sec. V that the probability matrix approximation leads to results concerning the mass splitting within the various strongly-interacting particle multiplets that are in rough agreement with experiment and with more detailed calculations in those cases that have been treated in detail. Application of the probability matrix approximation is extremely simple; practically none of the techniques of dispersion theory are used. Of course, it will be necessary to use more dispersion techniques than this to obtain accurate results. The main conclusion of the present paper is that the many-particle aspect of a universal bootstrap model, after it is understood, may lead to simplifications in the other aspects of the model. It may not be necessary to solve all the problems that plague present-day dispersion-theoretic treatments of systems of small numbers of particles.

In a universal bootstrap model each particle is a compound of itself and the other particles. It is often remarked that in such a model, no particle is truly "fundamental." However, it does not follow that there is no reasonable criterion that can distinguish certain particles as more fundamental than others. For example, it is reasonable to apply the classification scheme of Sec. II to the individual multiplets, and define those multiplets with σ-type mass deviations as fundamental. The approximate calculations of Sec. V indicate that the P-meson multiplet probably is fundamental according to this definition, the V and D probably are not fundamental, while the B multiplet may be fundamental. This definition of fundamentality corresponds with our intuitive ideas, since the P and B multiplets are the lightest of the boson and fermion multiplets. If we adopt the picture that only the P and B multiplets are fundamental, we would expect the fractional mass splitting to be greatest in these multiplets. (The reason for this effect is clear in the approximation used in this paper for the V and D multiplets, since the various particles in the nonfundamental multiplets are "averages" of the different pairs of particles in the fundamental multiplets.) Furthermore, it is not surprising in this picture if the Okubo sum rule is more nearly satisfied for the fundamental than for the nonfundamental multiplets.

APPENDIX: MODEL LEADING TO PROBABILITY MATRIX EQUATION

In this Appendix the probability matrix equation [Eq. (9) of Sec. IV] is derived in a simple, dispersion theoretic model. For definiteness we consider the baryon poles, and consider only coupled states of the (BP) type. The proton pole, for example, is coupled to the six states, $p\pi^0$, $n\pi^+$, $p\eta$, ΛK^+, $\Sigma^0 K^+$, and $\Sigma^+ K^0$. The scattering amplitude in the partial wave of the pole is a matrix T in the space of the coupled channels. At energies sufficiently high that all channels are open, T is related to the unitary matrix S by the equation, $T=(2i)^{-1}\theta^{-\frac{1}{2}}(S-1)\theta^{-\frac{1}{2}}$, where $\theta^{1/2}$ is the (non-negative definite) phase-space matrix, diagonal in the representation of the physical two-particle states. The amplitude T may be analytically continued to energies at which some or all of the channels are closed. As in Sec. II we make the simplifying assumption that the pole is a simple pole that occurs at an energy below all the two-particle thresholds.

The dispersion model used is the simple ND^{-1} model used in previous references.[4,29] In this model the numerator matrix N is computed from the Born approximation, and a once-subtracted dispersion relation is written for the denominator matrix D. In the probability matrix approximation, the effects of deviations from degeneracy on the left-hand cut are neglected, so that the numerator matrix is a constant matrix F multiplied by a function of energy, i.e., $N_{ij}=\beta(\omega)F_{ij}$. It is assumed that the basic force is "attractive," i.e., that $\beta(\omega)$ is positive for all values of ω above the lowest threshold, and that $\beta(\omega)$ is sufficiently well behaved at high energies so that the dispersion integrals converge.

The unitarity condition for D is Im $D=-\theta N=-\beta(\omega)\theta F$. The condition that a bound state occur is that the determinant of the denominator matrix vanish, i.e.,

$$|1-\int d\omega' g(\omega',\omega_0)(\theta F)|=0, \tag{A1}$$

where ω_0 is the bound-state energy. The exact nature of the function $g(\omega',\omega_0)$ is not important in the present discussion, but this function is independent of the deviations in the masses.

We define the k representation as a representation in which F is diagonal. Since the effects of mass splitting on the left-hand cut are neglected, the transformation matrix connecting this representation to the representation of the physical two-particle states is independent of the variations of the P and B masses. The nondiagonal elements of D in k representation are of

[29] F. Zachariasen and C. Zemach, Phys. Rev. 128, 849 (1962).

the form,

$$D_{kl}=-\int d\omega' g(\omega',\omega_0)\theta_{kl}F_{ll}.$$

Since θ is a multiple of the unit matrix in the degeneracy solution, the nondiagonal element θ_{kl} is of first order in the deviations from degeneracy. All terms in the determinant in Eq. (A1) that contain nondiagonal elements of D are of at least second order in these nondiagonal elements. Thus, to first order in the mass splitting, Eq. (A1) is a product of factors in the k representation, i.e.,

$$\Pi_k\left(1-\int d\omega' g(\omega',\omega_0)\theta_{kk}F_{kk}\right)=0. \tag{A2}$$

We denote by r the value of k corresponding to the factor that is zero. The matrix θ is diagonal in the physical (i) representation, so we express θ_{rr} in terms of the θ_{ii}, i.e., $\theta_{rr}=\Sigma_i A_{ri}{}^2\theta_{ii}$, where A_{ri} is the orthogonal transformation matrix connecting the i and k representations. It is easy to see that in the degeneracy limit, A diagonalizes T, so that $A_{ri}{}^2$ may be set equal to P_i, the probability of the state i in the resonating eigenstate of T. The bound-state condition may be written

$$1-\sum_i P_i I_i=0, \tag{A3}$$

$$I_i=\int_{q_i{}^2=0}^{\infty} d\omega' g(\omega',\omega_0)\theta_{ii}F_{rr}. \tag{A4}$$

If first-order deviations of the bound-state energy and of the squares of the masses are allowed, Eq. (A3) leads to the equation

$$\frac{dI_0}{d\omega_0}d\omega_0=-\left[\sum_i P_i\left(\frac{\partial I_i}{\partial \mu_i{}^2}d\mu_i{}^2+\frac{\partial I_i}{\partial m_i{}^2}dm_i{}^2\right)\right]_0, \tag{A5}$$

where the subscript 0 means that all quantities should be evaluated in the degeneracy-solution limit. The dependence of I_i on deviations of the $\mu_i{}^2$ and $m_i{}^2$ results entirely from the dependence of the phase-space factor θ_{ii} on these mass variables. Since θ_{ii} is an increasing function of the momentum in state i, and the quantities $g(\omega',\omega_0)$, θ_{rr}, and F_{rr} in Eq. (A4) are positive, the derivatives $\partial I_i/\partial \mu_i{}^2$ and $\partial I_i/\partial m_i{}^2$ are negative. The probability matrix approximation [interpretation of the quantities $\Pi_{B,ij}$ and $\Pi_{P,ij}$ in Eq. (7) as probabilities, together with the condition that α_B and α_P are positive], follows directly from Eq. (A5). The ratio of the quantities α_P and α_B of Eq. (7) is given by

$$\frac{\alpha_P}{\alpha_B}=\left(\mu_i{}^2\frac{\partial I_i}{\partial \mu_i{}^2}\right)\Big/\left(m_i{}^2\frac{\partial I_i}{\partial m_i{}^2}\right). \tag{A6}$$